Math 279 Lecture 8 Notes

Daniel Raban

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1 Solving ODEs Via Rough Integration

1.1 Solving for the Itô-Lyons map

We now turn to the ODE of the form

$$\dot{y} = \sigma(y)\dot{x}, \qquad y(0) = y^0,$$

where $x \in \mathcal{C}^\alpha$ and σ is sufficiently smooth. Here, $x : [0,T] \to \mathbb{R}^\ell$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times\ell}$, and $y : [0,T] \to \mathbb{R}^d$. We find a unique solution to this ODE, provided that we choose a suitable $\mathbb{X}$ so that $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}^\alpha$. The solution we come up with, $y(\cdot) = \mathscr{I}(y^0, \mathbf{x})$ is continuous (even locally Lipschitz) in y^0 and $\mathbf{x}$. $\mathscr{I}$ is known as the **Itô-Lyons map**. Let's make some preparations for this construction. Needless to say that we want to interpret this ODE as

$$y(t) = y^0 + \int_0^t \sigma(y(\theta))\, dx(\theta).$$

Though if $\alpha < 1/2$ (say $\alpha \in (1/3, 1/2]$), we need to lift both $\sigma(y)$ and x to $(\sigma(y), \widehat{\sigma})$, $(x, \mathbb{X})$ with $(\sigma(y), \widehat{\sigma}) \in \mathscr{G}^\alpha(x)$, $(x, \mathbb{X}) \in \mathscr{R}^\alpha$.

Recall that $\mathscr{G}^\alpha(x)$ consists of pairs $(z, \widehat{z})$ (where we intuitively think of $\widehat{z}$ as a "derivative" of z with respect to x) such that $z, \widehat{z} \in \mathcal{C}^\alpha$ and

$$[\![(z, \widehat{z})]\!]_{2\alpha} := \sup_{s\neq t} \frac{|z(t) - z(s) - \widehat{z}(s)(x(t) - x(s))|}{|t-s|^{2\alpha}} < \infty.$$

Indeed, from the integral formulation of this ODE, we expect that if y solves the equation, then $(y, \sigma(y)) \in \mathscr{G}^\alpha(x)$.

Theorem 1.1. *Let $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}^\alpha$ for $\alpha \in (1/3, 1/2]$, and assume $\sigma \in \mathcal{C}_b^3$ (bounded derivatives). Then for each y^0, there exists a path $y \in \mathcal{C}^\alpha$ such that $y(0) = y^0$, $(y, \sigma(y)) \in \mathscr{G}^\alpha(x)$, and*

$$y(t) = y^0 + \int_0^t \underbrace{(\sigma(y), \widehat{\sigma}(y))}_{\sigma} \cdot d\,\underbrace{(x, \mathbb{X})}_{\mathbf{x}}.$$

Here, $\widehat{\sigma}(y) = [\widehat{\sigma}^{ijk}(y)]$ *with*

$$\widehat{\sigma}^{ijk}(y) = \sum_{r=1}^{d} \sigma^{i,j}_{y_r}(y)\sigma^{rk}(y).$$

Moreover, $\mathscr{I}(y^0, \mathbf{x})$ *is Lipschitz with Lipschitz norm calculated in terms of* $\|\sigma\|_{\mathcal{C}^3}$ *and* $\|\mathbf{x}\|_{\alpha,2\alpha}$.

The idea is to start from $\mathbf{y} = (y, \widehat{y})$ and set

$$\mathcal{F}_{\mathbf{x}}(\widehat{y})(t) = \left(y^0 + \int_0^{\cdot} (\sigma(y), \widetilde{\sigma}(y,\widehat{y})) \cdot d(x, \mathbb{X}), \sigma(y)\right),$$

where $\widetilde{\sigma}(y,\widehat{y}) = [\widetilde{\sigma}^{ijk}(y,\widehat{y})]$, where

$$\widetilde{\sigma}^{ijk}(y,\widehat{y}) = \sum_{r=1}^{d} \sigma^{ij}_{y_r}(y)\widehat{y}^{rk}.$$

If $\widehat{y}$ is a fixed point of $\mathcal{F}$, then we are done because then the Gubinelli derivative of such $\mathbf{y}$ must be $\sigma(y)$.

1.2 Breakdown of the map $\mathcal{F}$

Let's understand $\mathcal{F}$ better: Throughout, $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}^\alpha$ is fixed.

Step 1: Recall that for $\mathbf{z} = (x, \widehat{z}) \in \mathscr{G}^\alpha(x)$, we can define $w(t) = \int_0^t \mathbf{z}\, dx$, which satisfies

$$|w(t) - w(s) - z(s)(x(t) - x(s)) - \widehat{z}(s)\mathbb{X}(s,t)| \le c_0([z]_\alpha [x]_\alpha + [\widehat{z}]_\alpha [\mathbb{X}]_{2\alpha})|t-s|^{3\alpha}.$$

This suggests $\mathcal{F}_{\mathbf{x}} : \mathscr{G}^\alpha(x) \to \mathscr{G}^\alpha(x)$ by $\mathcal{F}^0_{\mathbf{x}}(z,\widehat{z}) = (w, z)$. In fact, $\mathcal{F}^0$ is linear and

$$[\![\mathcal{F}^0_{\mathbf{x}}(\mathbf{y})]\!]_{\alpha,2\alpha} \le c_0 [\mathbf{x}]_{\alpha,2\alpha}[\mathbf{y}]_{\alpha,2\alpha}.$$

Here is the short proof of this:

Proof.

$$|w(t) - w(s) - z(s)(x(t) - x(s))| \le \|\widehat{z}\|_{L^\infty}[\mathbb{X}]_{2\alpha}|t-s|^{2\alpha}$$
$$+ c_0(\text{what we had before})|t-s|^{3\alpha}. \quad \square$$

Step 2: Define $\mathcal{F}^1_{\mathbf{x}} : \mathcal{G}^\alpha(x) \to \mathcal{G}^\alpha(x)$ with $\mathcal{F}^1_{\mathbf{x}}(z,\widehat{z}) = (\sigma(z), D\sigma(z)\widehat{z})$, where

$$(D\sigma(z)\widehat{z})^{ijk} = \sum_{r=1}^{d} \sigma^{ij}_{z_r}\widehat{z}^{rk}$$

and $\mathcal{F}^1$ is bounded if $\sigma \in \mathcal{C}^2$. Here is the proof:

Proof. Using a Taylor expansion for σ,

$$\begin{aligned}
&|\sigma(z(t)) - \sigma(z(s)) - D\sigma(z(s))\widehat{z}(s)x(s,t)| \\
&\qquad \leq |D\sigma(z(s))(z(t)-z(s)) - D\sigma(z(s))\widehat{z}(s)x(s,t)| + \|D^2\sigma\|_{L^\infty}[z]_\alpha|t-s|^{2\alpha} \\
&\qquad \leq \|D\sigma\|_{L^\infty}[\mathbf{z}]_{\alpha,2\alpha}|t-s|^{2\alpha} + \|D^2\sigma\|_{L^\infty}[z]_\alpha|t-s|^{2\alpha} \\
&\qquad \leq \|\sigma\|_{\mathcal{C}^2}[\mathbf{z}]_{\alpha,2\alpha}|t-s|^{2\alpha}.
\end{aligned}$$

So we get that

$$[\![\mathcal{F}^1_{\mathbf{x}}(\mathbf{z})]\!]_{\alpha,2\alpha} \leq \|\sigma\|_{\mathcal{C}^2}[\![\mathbf{z}]\!]_{\alpha,2\alpha}.$$

□

Step 3: Next, we define $\mathcal{F} : \mathscr{G}^\alpha(x) \to \mathscr{G}^\alpha(x)$, as $\mathcal{F} = \mathcal{F}^0 \circ \mathcal{F}^1$, so we send

$$(y, \widehat{y}) \mapsto (\sigma(y), D\sigma(y)\widehat{y}) \mapsto \left(\int_0^\cdot (\sigma\widehat{\sigma}) \cdot\, d(x, \mathbb{X}), \sigma(y)\right).$$

Then set

$$\mathcal{F}'(y, \widehat{y}) = \left(y^0 + \int_0^\cdot (\sigma, \widehat{\sigma}) \cdot\, d(x, \mathbb{X}), \sigma(y)\right).$$

We need to turn $\mathcal{F}'$ into a contraction so that it has a fixed point. We achieve this by choosing a sufficiently small interval $[0, t_0)$, and finding a nice invariant subset of $\mathscr{G}^\alpha(x)$. As we will see, t_0 depends on $\|\sigma\|_{\mathcal{C}^3}$, so we can repeat the same construction on $[t_0, 2t_0), \ldots$.

How can this be done? First, switch from $\mathscr{G}^\alpha(x)$ to $\widehat{\mathscr{G}^\alpha}(x) = \{(y, \widehat{y}) : y(0) = y^0, \widehat{y}(0) = \sigma(y^0)\}$. This way, we don't need to worry about the difference between a norm and a seminorm; this contraction takes place in a metric space, which is good enough for our purposes. Observe that $(a, \widehat{a}) \in \widehat{\mathscr{G}^\alpha}(x)$, where $a(t) = y^0 + \sigma(y^0)(x(t) - x(0))$ and $\widehat{a}(t) = \sigma(y^0)$. Observe that

$$\underbrace{a(t) - a(s)}_{\sigma(y^0)(x(t)-x(s))} - \underbrace{\widehat{a}(s)}_{\sigma(y^0)}(x(t) - x(s)) = 0.$$

Now set $\mathscr{B} = \{(y, \widehat{y}) \in \widehat{\mathscr{G}^\alpha}(x) : [\![(y - a, \widehat{y} - \widehat{a})]\!]_{\alpha,2\alpha} \leq 1\}$. The trick is to construct something in a rougher space and then show that it is as regular as you want. We will continue this next time.